THE MUTATION CLASS OF D_n QUIVERS

DAGFINN F. VATNE

ABSTRACT. We give an explicit description of the mutation classes of quivers of type D.

INTRODUCTION

In their very influential work on cluster algebras, Fomin and Zelevinsky defined the notion of *matrix mutation* [FZ1]. When applied to skew-symmetric matrices, this can be interpreted as an operation on quivers (i.e. directed graphs), and this is called *quiver mutation*. The quivers whose underlying graphs are simply laced Dynkin diagrams have a special significance in the cluster algebra theory, as they appear in the finite type classification [FZ2].

The purpose of this note is to give an explicit description of the *mutation class* of D_n quivers, for $n \geq 4$. That is, we will present the set of quivers which can be obtained by iterated mutation on a quiver whose underlying graph is of Dynkin type D. It turns out that the quivers in these mutation classes are easily recognisable. In particular, our result gives a complete description of the cluster-tilted algebras of type D, see [BMR1, BMR2, CCS2].

The shape of these quivers can be deduced from Schiffler's geometric model for the cluster categories of type D [Sch]. Nevertheless, it is convenient to have an explicit description. The method used in this paper to obtain the description is purely combinatorial, and no prerequisites are needed.

In order to understand these mutation classes, it is necessary to understand the mutation classes of quivers of Dynkin type A. These are explicitly described in [S] and [BV], and are also implicit in [CCS1]. They will be recalled in Section 2.

1. QUIVERS AND MUTATION

In this section we will briefly recall the definition of quiver mutation from [FZ1]. Before that, we fix some standard terminology about quivers which will be used here.

A *quiver* Q is a directed graph, that is, a quadruple (Q_0, Q_1, h, t) consisting of two sets Q_0 and Q_1, whose elements are called *vertices* and *arrows* respectively, and two functions $h, t : Q_1 \to Q_0$ (assigning a "head" and a "tail" to each arrow). We will often think of Q as the union of Q_0 and Q_1, and keep in mind that each element of Q_1 connects two vertices and has a direction. If either $h(\alpha) = i$ or $t(\alpha) = i$, we say that α is *incident* with i. Moreover, if $t(\alpha) = j$ and $h(\alpha) = i$, we will say that α is an arrow *from j to i*.

For any vertex i in Q_0, the *valency* of i (in Q) is the number of neighbouring vertices, i.e. the number of vertices $j \neq i$ such that there exists an arrow $\alpha \in Q_1$ with either $h(\alpha) = i$ and $t(\alpha) = j$ or vice versa.

We will assume throughout that quivers do not have loops or 2-cycles. In other words, for any $\alpha \in Q_1$, we have that $h(\alpha) \neq t(\alpha)$ and there does not exist $\beta \in Q_1$ such that $h(\beta) = t(\alpha)$ and $t(\beta) = h(\alpha)$.

For a quiver $Q = (Q_0, Q_1, h, t)$ and a vertex $i \in Q_0$, the quiver $\mu_i(Q) = (Q_0^*, Q_1^*, h^*, t^*)$ is obtained by making the following changes to Q:

- All arrows incident with i are reversed, i.e. $h^*(\alpha) = t(\alpha)$ and $t^*(\alpha) = h(\alpha)$ for such arrows.
- Whenever $j, k \in Q_0$ are such that there are $r > 0$ arrows from j to i (in Q) and $s > 0$ arrows from i to k (in Q), first add rs arrows from j to k. Then remove a maximal number of 2-cycles.

There is a choice involved in the last step in this procedure, but this will not concern us, as the possible resulting quivers are isomorphic as quivers, and will be regarded as equal. The quiver $\mu_i(Q)$ so defined is said to be the quiver obtained from Q by *mutation* at i.

Mutation at a vertex is an involution, that is, $\mu_i(\mu_i(Q)) = Q$. It follows that mutation generates an equivalence relation on quivers. Two quivers are *mutation equivalent* if one can be obtained from the other by some sequence of mutations. An equivalence class will be called a *mutation class*.

The following lemma is well known and easily verified:

Lemma 1.1. *If the quivers Q_1 and Q_2 both have the same underlying graph T, and T is a tree, then Q_1 and Q_2 are mutation equivalent.*

In particular, by this lemma, it makes sense to speak of the mutation classes of the simply laced Dynkin diagrams. In this paper we will be concerned with the mutation classes of type A_k quivers:

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow (k-1) \longrightarrow k \tag{1}$$

and with the mutation classes of type D_n quivers:

$$\begin{tikzcd}[column sep=small] & & & & & (n-1) \\ 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & (n-2) \arrow[ur] \arrow[dr] & \\ & & & & & n \end{tikzcd} \tag{2}$$

By the cluster algebra finite-type classification [FZ2], we know that these mutation classes are finite.

2. The set $\mathcal{M}_n^D$

Let first $\mathcal{M}_k^A$ be the mutation class of A_k. This set of quivers consists of the connected quivers that satisfy the following [BV]:

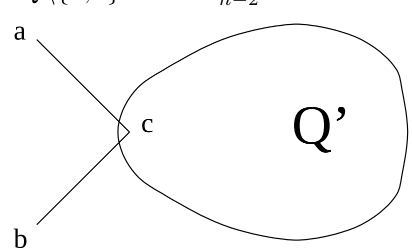
- there are k vertices,
- all non-trivial cycles are oriented and of length 3,
- a vertex has valency at most four,
- if a vertex has valency four, then two of its adjacent arrows belong to one 3-cycle, and the other two belong to another 3-cycle,
- if a vertex has valency three, then two of its adjacent arrows belong to a 3-cycle, and the third arrow does not belong to any 3-cycle

By a *cycle* in the first condition we mean a cycle in the underlying graph, not passing through the same edge twice. The union of all $\mathcal{M}_k^A$ for all k will be denoted by $\mathcal{M}^A$. Note that for a quiver Γ in $\mathcal{M}^A$, any connected subquiver of Γ is also in $\mathcal{M}^A$.

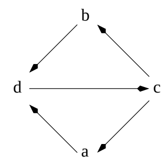
For a quiver Γ in $\mathcal{M}^A$, we will say that a vertex v is a *connecting* vertex if v has valency at most 2 and, moreover, if v has valency 2, then v is a vertex in a 3-cycle in Γ.

We now define a class $\mathcal{M}_n^D$ of quivers which will be shown to be the mutation class of D_n. We define $\mathcal{M}_n^D$ to be the set of quivers Q with n vertices belonging to one of the following four types:

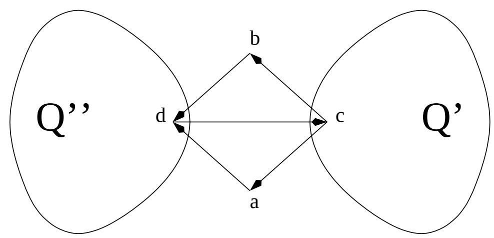
Type I: Q has two vertices a and b which have valency one and both a and b have an arrow to or from the same vertex c, and $Q' = Q \setminus \{a, b\}$ is in $\mathcal{M}_{n-2}^A$ and c is a connecting vertex for Q'.



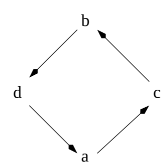
Type II: Q has a full subquiver Q_1 with four vertices which looks like



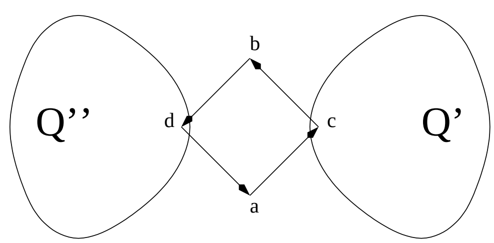
where the vertices a and b have valency 2 in Q, and $Q \backslash \{a, b, d \to c\} = Q' \dot\cup Q''$ is disconnected with two components Q' and Q'' which are both in $\mathcal{M}^A$ and for which c and d are connecting vertices.



Type III: Q has a full subquiver which is a directed 4-cycle:

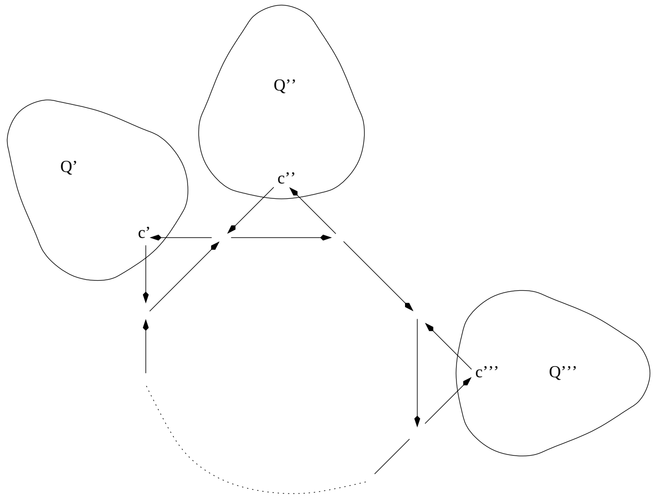


and $Q \backslash \{a, b\} = Q' \dot\cup Q''$ is disconnected with two components Q' and Q'' which are both in $\mathcal{M}^A$ and for which c and d are connecting vertices, like for Type II.



Type IV: Q has a full subquiver which is a directed k-cycle, where $k \geq 3$. We will call this the *central* cycle. For each arrow $\alpha : a \to b$ in the central cycle, there may (and may not) be a vertex c_α which is not on the central cycle, such that there is an oriented 3-cycle traversing $a \xrightarrow{\alpha} b \to c_\alpha \to a$. Moreover, this 3-cycle is a full subquiver. Such a 3-cycle will be called a *spike*. There are no more arrows starting or ending in vertices on the central cycle.

Now $Q \backslash \{\text{vertices in the central cycle and their incident arrows}\} = Q' \dot\cup Q'' \dot\cup Q''' \dot\cup \cdots$ is a disconnected union of quivers, one for each spike, which are all in $\mathcal{M}^A$, and for which the corresponding vertex c is a connecting vertex:



Remark 2.1. One should note that in Types II, III and IV, the subquivers $Q', Q'', \ldots$ can be in the set $\mathcal{M}_1^A$, i.e. they can have only one vertex.

Remark 2.2. Ringel [R] has described all self-injective cluster tilted algebras, and in particular shown that they are of type D. Their Gabriel quivers are all Type IV in our description; they are the special cases with either no spikes, or with a maximal number of spikes and with the subquivers $Q', Q'', \ldots$ with only one vertex.

3. Proof that $\mathcal{M}_n^D$ is the mutation class of D_n

The main result of this paper is the following:

Theorem 3.1. *For any $n \geq 4$, a quiver Q is mutation equivalent to D_n if and only if it is in $\mathcal{M}_n^D$.*

We will break the proof of Theorem 3.1 into small lemmas. The first takes care of the type A pieces.

Lemma 3.2. *Let $\Gamma \in \mathcal{M}_k^A$ for some $k \geq 2$, and let c be a connecting vertex for Γ. Then there exists a sequence of mutations on Γ satisfying the following:*

- *μ_c does not appear in the sequence*
- *the resulting quiver is isomorphic with the quiver in* (1)
- *under this isomorphism, c is relabelled as 1*

Proof. We will prove the claim by induction on k. It is readily checked for small values.

Assume first that c has valency 1. Noting that the neighbour c' of c is a connecting vertex for the quiver $\Gamma' = \Gamma \backslash \{c\}$, we have by induction that Γ' can be mutated to a linearly oriented A_{k-1} quiver without mutating at c'. The result looks like this after the relabelling:

$$c \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow (k-2) \longrightarrow (k-1)$$

If the edge between c and 1 is correctly oriented, we are done. If not, perform the sequence of mutations $\mu_1, \mu_2, \ldots, \mu_{(k-2)}, \mu_{(k-1)}$ and relabel to get the quiver in (1).

Assume now the other possibility, namely that c has valency 2. It is then traversed by a 3-cycle. Again we note that the neighbouring vertices c' and c'' are connecting vertices for their respective components of the quiver $\Gamma' = \Gamma \backslash \{c, c' \to c''\}$, we can apply the induction hypothesis and perform a sequence of mutations to produce the following quiver:

$$\begin{tikzcd}
 & c' \arrow[r] \arrow[dl] & 1' \arrow[r] & 2' \arrow[r] & \cdots \arrow[r] & k_1' \\
c \arrow[dr] & & & & & \\
 & c'' \arrow[uu] \arrow[r] & 1'' \arrow[r] & 2'' \arrow[r] & \cdots \arrow[r] & k_2''
\end{tikzcd}$$

Now the sequence $\mu_{c'}, \mu_{1'}, \mu_{2'}, ..., \mu_{k'_1}$ and a relabelling will yield the desired result. $\square$

Lemma 3.3. *All quivers of Type I are mutation equivalent to D_n quivers.*

Proof. If we have a quiver Q as in the description of Type I, then c is connecting for Q'. An application Lemma 3.2 shows that we can mutate Q' to a quiver with underlying graph A_{n-2} by mutating in vertices not equal to c. This induces a mutation of Q into a quiver with underlying graph D_n. $\square$

Lemma 3.4. *All quivers of Type II are mutation equivalent to D_n quivers.*

Proof. Let Q be a quiver as in the description of Type II. The vertices c and d are connecting for Q' and Q'', so we can apply Lemma 3.2 to each of these. Thus we can mutate Q into the following quiver, for some $0 \leq m \leq n-4$:

$$\begin{tikzcd}
 & & & & b \arrow[dl] & & & & \\
1 & \cdots \arrow[l] & m \arrow[l] & d \arrow[l] \arrow[rr] & & c \arrow[ul] \arrow[dl] \arrow[r] & (m+1) \arrow[r] & \cdots \arrow[r] & (n-4) \\
 & & & & a \arrow[ul] & & & &
\end{tikzcd}$$

The sequence $\mu_d, \mu_m, \mu_{(m-1)}, ..., \mu_1$ of mutations will now result in a quiver with underlying graph D_n. $\square$

Lemma 3.5. *All quivers of Type III are mutation equivalent to D_n quivers.*

Proof. Let Q be any quiver Q as in the description of Type III. If we mutate at the vertex a, we get a quiver $\mu_a(Q)$ of Type II. By Lemma 3.4, we have that $\mu_a(Q)$ is mutation equivalent to D_n, and therefore so is Q. $\square$

Lemma 3.6. *The oriented cycle of length n is mutation equivalent to D_n.*

Proof. Starting with the quiver (2), we get an oriented cycle by performing the following sequence of mutations: $\mu_{(n-1)}, \mu_{(n-2)}, ..., \mu_1$. $\square$

Lemma 3.7. *All quivers of Type IV are mutation equivalent to D_n quivers.*

Proof. Let Q be a quiver as in the description of Type IV. By Lemma 3.6 it is sufficient to show that Q is mutation equivalent to an oriented cycle.

Let $a \to b \to c \to a$ be a spike in Q with $a \to b$ on the central cycle, and let Q_c be the corresponding type A piece. Since c is connecting for Q_c, we can mutate Q_c (without mutating at c) to a linearly oriented A_k quiver, for some k. This induces an iterated mutation of Q resulting in a quiver with a full subquiver looking like this:

$$\begin{tikzcd}
a \arrow[dd] & & & & & \\
 & c \arrow[ul] \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & k \\
b \arrow[ur] & & & & &
\end{tikzcd}$$

Performing the mutations $\mu_c, \mu_1, \mu_2, ..., \mu_k$ on this quiver yields a quiver which is just a directed path from a to b. This induces an iterated mutation on Q which in effect replaces the spike involving c and the corresponding type A subquiver Q_c with a directed path from a to b.

Doing this to all the spikes and the type A pieces, we get an oriented n-cycle, which is what we wanted. $\square$

We now put the pieces together to prove the main result:

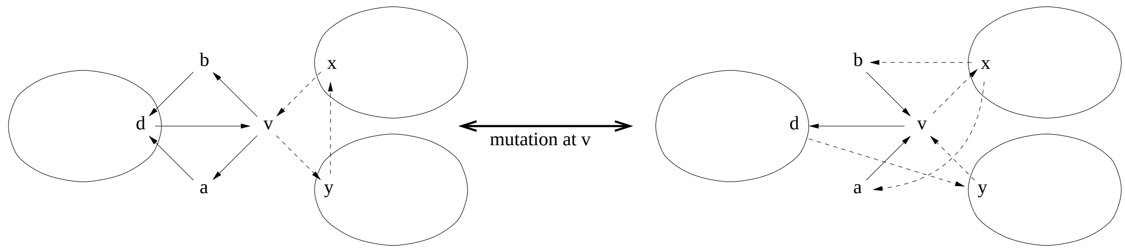


FIGURE 1. Mutation of a Type II quiver. The blobs represent type A subquivers, and the dotted arrows indicate that they may not appear. If the x-blob is not there, $\mu_v(Q)$ is Type I. Otherwise, it is Type II.

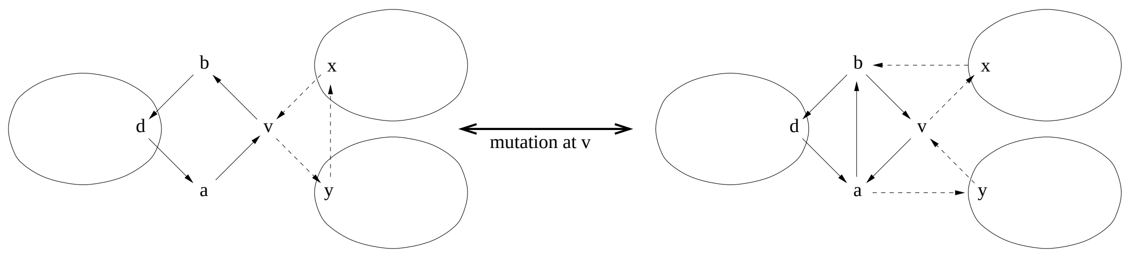


FIGURE 2. If we mutate at v, the Type III quiver Q on the left is mutated to the Type IV quiver $\mu_v(Q)$ on the right, where the central cycle has length 3.

Proof of Theorem 3.1. By the lemmas above, all quivers in $\mathcal{M}_n^D$ are mutation equivalent to D_n. It remains to show that $\mathcal{M}_n^D$ is closed under mutation.

Let Q be some quiver in $\mathcal{M}_n^D$ and let v be some vertex. If v is inside one of the type A pieces $Q', Q'', ...$ as in the descriptions in Section 2, but not the connecting vertex connecting the piece to the rest of the quiver Q, then the mutated quiver $\mu_v(Q)$ is still of the same type. It therefore suffices to check what happens when we mutate at the other vertices, and we will consider four cases, according to the type of Q.

Case I. Suppose that Q is of Type I. If v is one of the vertices a, b (as in the description), then $\mu_a(Q)$ is obviously also of Type I, as μ_v only reverses the incident arrow. Assume therefore that v is the vertex c, connecting for Q'. If v has valency 2 in Q', then the quiver $\mu_v(Q)$ is of Type II or Type IV (with a central cycle of length 3), depending on the orientation of the arrows between v and a, b. If v has valency 1 in Q', then the quiver $\mu_v(Q)$ is either of Type I (if v is a source or a sink in Q), Type II or Type IV (with a central cycle of length 3), depending on the orientations.

Case II. Let now Q be of Type II. For v equal to a or b in the description of the type, we have that $\mu_v(Q)$ is of Type III. If v is c and the valency of v in Q' is 2, then then $\mu_v(Q)$ is also of Type II. If the valency is 1, then $\mu_v(Q)$ is either of Type I or Type II, depending on the orientation of the arrow incident with v in Q'. See Figure 1. The case when v is d is similar.

Case III. Suppose Q is of Type III. If v is one of the vertices a, b in the description, then $\mu_v(Q)$ is of Type II. So assume v is c. (The case of v equal to d is the same.) Then $\mu_v(Q)$ is of Type IV with a central cycle of length 3. See Figure 2.

Case IV. Finally, let Q be a quiver of Type IV. First consider a vertex v which is on the central cycle. If the length of the central cycle is 3, then $\mu_v(Q)$ is of Type III if the arrow on the opposite side of the central cycle is part of a spike and Type I if not. See Figure 3. (Recall our assumtion that $n \geq 4$.) If the length of the central cycle is 4 or more, then $\mu_v(Q)$ is also of Type IV, and has a central cycle one arrow shorter than for Q.

If v is a vertex on a spike, but not on the central cycle, then $\mu_v(Q)$ is also Type IV, with a central cycle one arrow longer than for Q.

We have now seen that for any quiver $Q \in \mathcal{M}_n^D$ and any vertex v, the mutated quiver $\mu_v(Q)$ is also in $\mathcal{M}_n^D$, so the proof is finished. $\square$

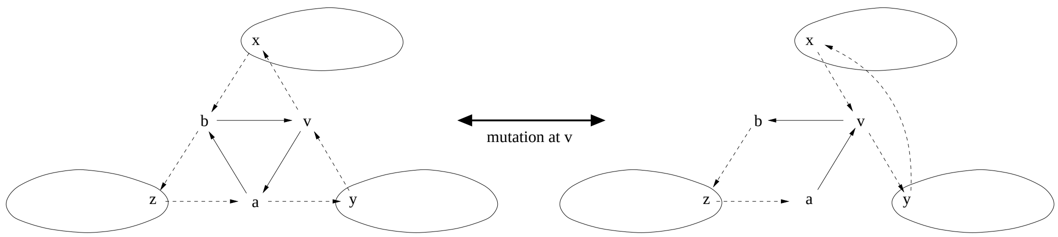


FIGURE 3. If Q (on the left) is Type IV with a central cycle of length 3, then $\mu_v(Q)$ is Type I or III, depending on whether there is an opposite spike.

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References

[BMR1] A. B. Buan, R. Marsh, I. Reiten *Cluster-tilted algebras*, Trans. Amer. Math. Soc. 359 no. 1 (2007), 323-332

[BMR2] A. B. Buan, R. Marsh, I. Reiten *Cluster-tilted algebras of finite representation type*, J. Algebra 306 no. 2 (2006), 412-431

[BV] A. B. Buan, D. Vatne *Derived equivalence classification of cluster-tilted algebras of type A_n*, J. Algebra 319 no. 7 (2008), 2723-2738

[CCS1] P. Caldero, F. Chapoton, R. Schiffler *Quivers with relations arising from clusters (A_n case)*, Trans. Amer. Math. Soc. 358, no. 3 (2006), 1347-1364

[CCS2] P. Caldero, F. Chapoton, R. Schiffler *Quivers with relations and cluster tilted algebras*, Algebr. Represent. Theory 9 (2006), 359-376

[FZ1] S. Fomin, A. Zelevinsky *Cluster Algebras I: Foundations*, J. Amer. Math. Soc. 15, no. 2 (2002), 497-529

[FZ2] S. Fomin, A. Zelevinsky *Cluster algebras II: Finite type classification*, Invent. Math. 154, no. 1 (2003), 63-121

[R] C. M. Ringel *The Self-injective Cluster Tilted Algebras*, preprint arXiv:0705.3903 (2007)

[Sch] R. Schiffler *A geometric model for cluster categories of type D_n*, J Algebr Comb 27, no. 1 (2008), 1-21

[S] A. Seven *Recognizing cluster algebras of finite type*, Electron. J. Combin. 14, no. 1 (2007), Research Paper 3, 35 pp. (electronic)

Institutt for matematiske fag, Norges teknisk-naturvitenskapelige universitet, N-7491 Trondheim, Norway

E-mail address: `dvatne@math.ntnu.no`